Estimation of Mixed Venous CO₂ Tension and QRS Electrical Axis From Simple Mathematical Considerations

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Abstract: Simple mathematical formulations are introduced to obtain mixed venous carbon dioxide tension ($P_{\text{VCO}_2}$) and mean electrical axis of the ventricles ($\theta$). A linear decrease in $O_2$ and an exponential increase in $CO_2$ concentrations of alveolar gas during rebreathing are shown to be essential for the estimation of $P_{\text{VCO}_2}$. The QRS electrical axis was usually obtained from a graphical analysis using Einthoven's triangle with peak height differences between the R and S waves. The $\theta$ value is calculated from a simple trigonometric function instead of the graphical analysis. These examples indicate that elementary mathematics is useful in physiology to make clear physiological meaning behind observed phenomena.

Key words: elementary mathematics, mixed venous CO₂ tension, QRS electrical axis

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Introduction

For the evaluation of cardiac output with direct Fick's method, mixed venous CO₂ tension ($P_{\text{VCO}_2}$) is necessary together with arterial CO₂ tension ($P_{\text{ACO}_2}$). $P_{\text{ACO}_2}$ can be measured directly by taking arterial blood. Alveolar $P_{\text{CO}_2}$ ($P_{\text{ACO}_2}$) is also a good approximation of $P_{\text{ACO}_2}$. However, $P_{\text{VCO}_2}$ cannot be measured easily because of the sampling difficulty of mixed venous blood. Indirect estimation of $P_{\text{VCO}_2}$ has been studied by a rebreathing method (Defaures, 1958; Mochizuki et al., 1984; Uchida et al., 1986; Vanhees et al., 2000). A rebreathing system is generally composed of the lung and a bag containing the air. Neglecting a dead space compared with a tidal volume, we can approximate $P_{\text{CO}_2}$ in the bag during rebreathing as $P_{\text{ACO}_2}$. Since the rebreathing system is a closed system, CO₂ is accumulated and $P_{\text{ACO}_2}$ rises. With the progress of the $P_{\text{ACO}_2}$ increase, diffusion of CO₂ from the mixed venous blood to alveoli is reduced, and finally the diffusion is stopped when $P_{\text{VCO}_2} = P_{\text{ACO}_2}$. Therefore, we can get an information on the blood ($P_{\text{VCO}_2}$) from that on the gas ($P_{\text{ACO}_2}$) without taking the blood. In this report fundamental relations necessary to estimate $P_{\text{VCO}_2}$ from the $O_2$ and CO₂ concentrations during rebreathing are shown.

The standard mean electrical axis of the ventricles ($\theta$) is 59 degrees, which is changed markedly in certain pathological conditions (Guyton, 1976). The $\theta$ value was usually determined from a graphical analysis of the standard leads electrocardiogram (ECG) using Einthoven's triangle. Projections of differences in peak heights between the R and S waves on the axes of leads I and III schematically give the $\theta$ value. Instead of the graphical analysis, $\theta$ is here shown to be obtained from an elementary mathematical consideration.

Estimation of $P_{\text{CO}_2}$ of the mixed venous blood

$P_{\text{ACO}_2}$ rises exponentially during rebreathing (Defaures, 1958), while alveolar $P_{\text{O}_2}$ ($P_{\text{AO}_2}$) is decreased linearly (Mochizuki et al., 1984; Uchida

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et al., 1986). Such a contrast in O\textsubscript{2} and CO\textsubscript{2} concentrations during rebreathing is due to the differences in pressure gradient between the alveolar gas and the mixed venous blood. The pressure gradient of CO\textsubscript{2} is about one tenth of that of O\textsubscript{2}, and the equilibration between \( P_{\text{A}CO_2} \) and \( P\tilde{V}CO_2 \) is attained within a contact time of the mixed venous blood with the alveolar gas. According to these observations, time dependence of the O\textsubscript{2} and CO\textsubscript{2} concentrations during rebreathing can be written as

\[ F(t) = a - bt \]

\[ G(t) = G_0 - (G_0 - G_b) \exp(-kt) \]

where \( F(t) \) and \( G(t) \) are time dependent concentrations of O\textsubscript{2} and CO\textsubscript{2} in a rebreathing gas, and \( a, b \) and \( k \) are positive constants (Fig.1). \( G_b \) and \( G_0 \) are CO\textsubscript{2} concentrations at the start and the end of rebreathing, respectively. Time dependent alveolar O\textsubscript{2} and CO\textsubscript{2} pressures are given by

\[ P_{\text{A}O_2}(t) = (P_b - 47) F(t) \]

\[ P_{\text{A}CO_2}(t) = (P_b - 47) G(t) \]

where \( P_b \) is a barometric pressure and 47 Torr is saturated water vapor pressure at 37 °C. At the start of rebreathing \( (t = 0) \), \( P_{\text{A}O_2}(0) \) and \( P_{\text{A}CO_2}(0) \) are calculated from the initial alveolar O\textsubscript{2} and CO\textsubscript{2} concentrations \( (F_0 \) and \( G_0) \). At the end of the rebreathing \( (t \rightarrow \infty) \), the limiting \( P_{\text{A}CO_2}(t) \), which is equal to \( G_0 \) corresponds to \( P\tilde{V}CO_2 \) because of the equilibration of CO\textsubscript{2} tension between the alveolar gas and the mixed venous blood.

Neglecting the lung volume change during rebreathing, we have \( \dot{V}_0 \) and \( \dot{V}CO_2 \) as follows:

\[ \dot{V}_0 = -V(dF(t)/dt) = Vb \]

\[ \dot{V}CO_2 = V(dG(t)/dt) =Vk(G_0 - G_b) \exp(-kt) \]

Equation (5) shows that \( \dot{V}_0 \) remains constant even in rebreathing, and Eq. (6) shows that \( \dot{V}CO_2 \) is decreased exponentially with the time. From Eqs. (5) and (6), respiratory quotient \( (RQ) \) is given by

\[ RQ = (k/b) (G_0 - G_b) \exp(-kt) \]

which is rewritten with Eq. (2) as

\[ RQ = (k/b) \{G_0 - G(t)\} \]

Substituting \( G(t) \) in Eq. (4) for that in Eq. (8), we have

\[ RQ = -(k/b) P_{\text{A}CO_2}(t)/(P_b - 47) + (k/b)G_0 \]

Decreasing \( \dot{V}CO_2 \) and constant \( \dot{V}_0 \) during rebreathing give rise to a linear reduction of \( RQ \) against the \( P_{\text{A}CO_2}(t) \) increase (Eq. (9)). The limiting value \( G_0 \) is equal to the \( G(t) \) value when \( RQ = 0 \), reflecting the fact that at this stage no CO\textsubscript{2} output occurs because of the disappearance of pressure gradient between alveoli and mixed venous blood. Therefore,

\[ P\tilde{V}CO_2 = (P_b - 47) G_0 = (P_b - 47) G(t)_{RQ=0} \]

Mochizuki et al. (1984) developed a method to obtain \( P\tilde{V}CO_2 \) from a linear relation between \( P_{\text{A}CO_2} \) and \( RQ \) during rebreathing. The above discussion shows that their method is based on the fundamental two relations given by Eqs. (1) and (2).

**Estimation of the QRS electrical axis**

The QRS electrical axis (\( \theta \)) of ECG was usually obtained using the differences in peak heights between R and S waves for the leads I and III (Fig.2). In this figure, O is the middle point of Einthoven's triangle, and A is an intersection point with the projections of leads I and III. Segments OB and AC correspond to the peak height differences between the R and S waves for the leads I and III, and are represented here I and III, respectively. The \( \theta \) value is obtained by measuring the angle \( \angle BOA \). It should be noted that \( \angle BOC = \angle BAC = 30^\circ \), and therefore \( \angle OAC = \angle OAB + \angle BAC = (90^\circ - \theta) + 30^\circ = 120^\circ - \theta \). Referring to the two triangles \( \Delta OAB \) and \( \Delta OAC \) in this figure, we have

\[ OA \cos \theta = OB = I \]

\[ OA \cos(120^\circ - \theta) = AC = III \].

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Fig. 2  Relation between the QRS electrical axis and peak heights of the standard leads ECG

From Eqs. (11) and (12)

\[-\frac{1}{2} \cdot \cos \theta + \frac{\sqrt{3}}{2} \cdot \sin \theta \] / \cos \theta = \frac{\text{III}}{\text{I}} (13)

Therefore,

\[\tan \theta = 2/\sqrt{3} \left( \frac{\text{III}}{\text{I}} + 1/2 \right) \] (14)

The QRS electrical axis \(\theta\) can be calculated as follows without drawing a diagram like Fig. 2:

\[\theta = \tan^{-1} \left( \frac{2}{\sqrt{3}} \left( \frac{\text{III}}{\text{I}} + 1/2 \right) \right) \] (15)

References

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